

Category Theory

Functors and Natural Transformations

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Recap

A category \mathcal{C} consists of:

- ▶ A collection of objects: A, B, C, \dots
- ▶ For each pair of objects, a collection of morphisms: $\mathbf{Hom}(A, B)$
- ▶ A composition operation: given $f : A \rightarrow B$ and $g : B \rightarrow C$, a morphism $g \circ f : A \rightarrow C$
- ▶ For each object A , an identity morphism $\mathbf{id}_A : A \rightarrow A$

Subject to:

- ▶ Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$
- ▶ Identity laws: $f \circ \mathbf{id}_A = f = \mathbf{id}_B \circ f$

Functors

Mapping between Categories that conserves structure

A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ assigns:

- ▶ To each object $A \in \mathcal{C}$, an object $F(A) \in \mathcal{D}$
- ▶ To each morphism $f : A \rightarrow B$, a morphism $F(f) : F(A) \rightarrow F(B)$

Such that:

- ▶ $F(\text{id}_A) = \text{id}_{F(A)}$ (preserves identity)
- ▶ $F(g \circ f) = F(g) \circ F(f)$ (preserves composition)

Cat and Functor Categories

The category **Cat**:

- ▶ Objects: (small) categories
- ▶ Morphisms: functors between categories
- ▶ Composition: functor composition $(G \circ F)$, identity: $\mathbf{Id}_{\mathcal{C}}$

The functor category $[\mathcal{C}, \mathcal{D}]$ (also written $\mathcal{D}^{\mathcal{C}}$):

- ▶ Objects: functors $F : \mathcal{C} \rightarrow \mathcal{D}$
- ▶ Morphisms: natural transformations $\alpha : F \Rightarrow G$
- ▶ Composition: given $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$, define $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$
- ▶ Identity: $(\mathbf{id}_F)_A = \mathbf{id}_{F(A)}$