# Category Theory

## Functors and Natural Transformations

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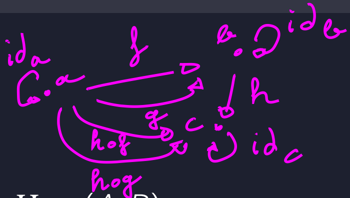
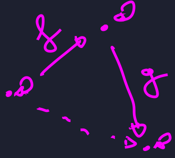



A category $\mathcal{C}$ consists of:

- A collection of objects: $A, B, C, \dots$
- For each pair of objects, a collection of morphisms: $\mathbf{Hom}(A, B)$
- A composition operation: given $f : A \to B$ and $g : B \to C$, a morphism $g \circ f : A \to C$
- For each object $A$, an identity morphism $\mathbf{id}_A : A \to A$

Subject to:

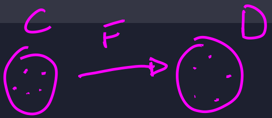
- Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$
- Identity laws: $f \circ \mathbf{id}_A = f = \mathbf{id}_B \circ f$




$id \circ f$

$g \circ f$

# Functors




Mapping between Categories that conserves structure

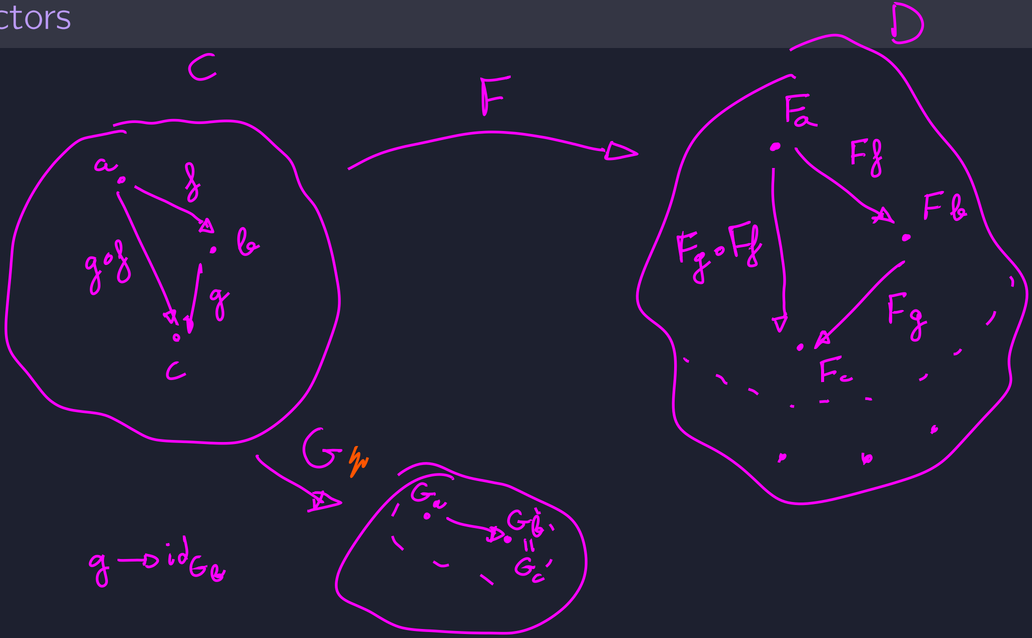
A functor $F : \mathcal{C} \to \mathcal{D}$ assigns:

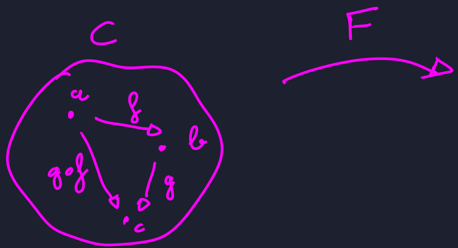
- To each object $A \in \mathcal{C}$, an object $F(A) \in \mathcal{D}$
- To each morphism $f : A \to B$, a morphism $F(f) : F(A) \to F(B)$

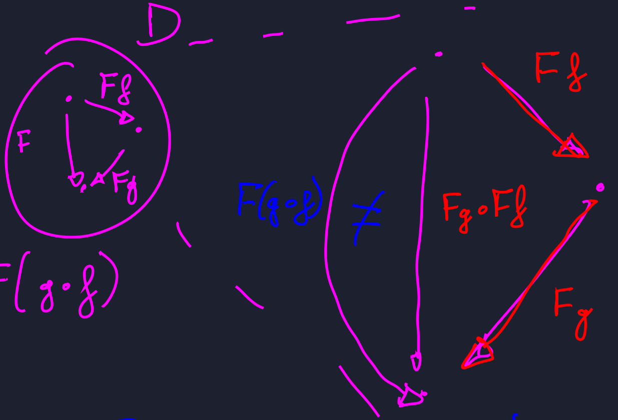
Such that:

- $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ (preserves identity)
- $F(g \circ f) = F(g) \circ F(f)$ (preserves composition)

# Functors



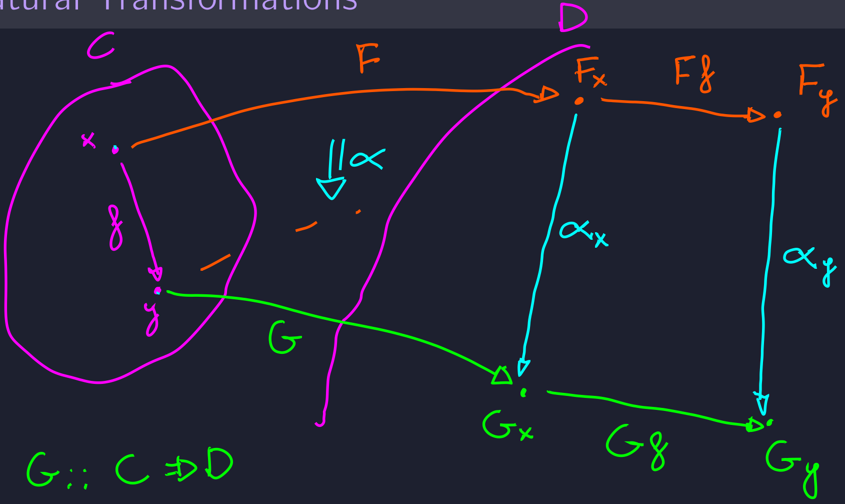



$$F_g \circ F_f \neq F(g \circ f)$$

F is not Functor

# Natural Transformations




$$Gf \circ \alpha_x = \alpha_y \circ Ff$$

$G :: C \Rightarrow D$

$F :: C \Rightarrow D$

$\alpha : F \Rightarrow G$

# Vertical Composition

$$
\begin{tikzcd}[row sep=large]
C \arrow[r, bend left=70, "F"{name=F}] \arrow[r, "G"{name=G, description}] \arrow[r, bend right=70, "H"'{name=H}] & D
\arrow[from=F, to=G, Rightarrow, "\alpha"]
\arrow[from=G, to=H, Rightarrow, "\beta"]
\end{tikzcd}
$$

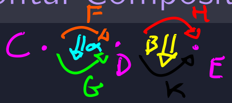
$$
C \quad
\begin{tikzcd}
x \arrow[d] \\
y
\end{tikzcd}
$$

$$
D \quad
\begin{tikzcd}[row sep=large, column sep=huge]
Fx \arrow[r, "Ff"] \arrow[d, "\alpha_x"] \arrow[dd, bend right=60, "\beta_x \circ \alpha_x"'] & Fy \arrow[d, "\alpha_y"'] \arrow[dd, bend left=60, "\beta_y \circ \alpha_y"] \\
Gx \arrow[r, "Gf"] \arrow[d, "\beta_x"] & Gy \arrow[d, "\beta_y"'] \\
Hx \arrow[r, "Hf"'] & Hy
\end{tikzcd}
$$

$$\beta \circ \alpha :: F \Rightarrow H$$

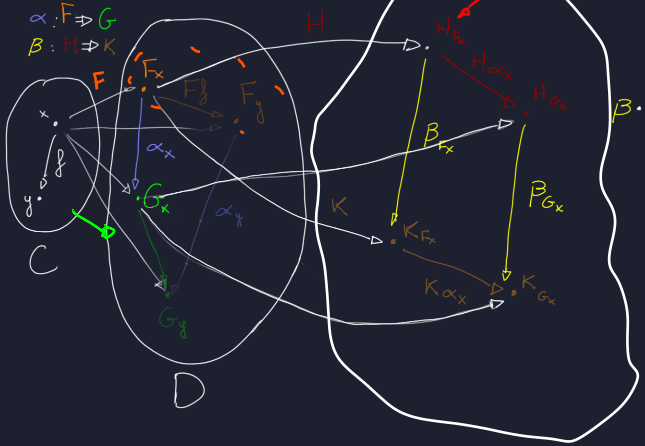
$$\begin{tikzcd}
C \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \arrow[r, bend left=50, "H"] \arrow[r, bend right=50, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & E
\end{tikzcd}$$

# Horizontal Composition

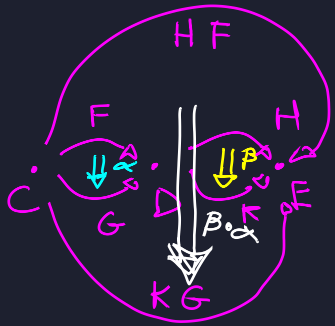



$\alpha : F \Rightarrow G$

$\beta : H \Rightarrow K$




$\beta \bullet \alpha = K_{\alpha_x} \circ \beta_{F_x} = \beta_{G_x} \circ H_{\alpha_x}$



# Interchange Law

$$\begin{tikzcd}[column sep=huge]
C \arrow[r, bend left=60, ""{name=U1, below}] \arrow[r, ""{name=M1, above}, ""{name=M1b, below}] \arrow[r, bend right=60, ""{name=L1, above}] \arrow[from=U1, to=M1, Rightarrow, "\alpha"] \arrow[from=M1b, to=L1, Rightarrow, "\alpha'"] & D \arrow[r, bend left=60, ""{name=U2, below}] \arrow[r, ""{name=M2, above}, ""{name=M2b, below}] \arrow[r, bend right=60, ""{name=L2, above}] \arrow[from=U2, to=M2, Rightarrow, "\beta"] \arrow[from=M2b, to=L2, Rightarrow, "\beta'"] & E
\end{tikzcd}$$

$$(\alpha' * \alpha) \circ (\beta' * \beta) = (\beta' \circ \alpha') * (\beta \circ \alpha)$$

# **Cat** and Functor Categories

$$\begin{tikzcd} A \arrow[r, bend left, "F"] & B \arrow[l, bend left, "G"] \end{tikzcd}$$

$F \circ G = Id_B$

$G \circ F = Id_A$

The category **Cat**:

- Objects: (small) categories
- Morphisms: functors between categories
- Composition: functor composition $(G \circ F)$, identity: $\mathbf{Id}_{\mathcal{C}}$

The functor category $[\mathcal{C}, \mathcal{D}]$ (also written $\mathcal{D}^{\mathcal{C}}$):

- Objects: functors $F : \mathcal{C} \to \mathcal{D}$
- Morphisms: natural transformations $\alpha : F \Rightarrow G$
- Composition: given $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$, define $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$
- Identity: $(\mathbf{id}_F)_A = \mathbf{id}_{F(A)}$