

# Category Theory

From objects and arrows to Algebra

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# Category

- ▶ Objects
- ▶ Morphisms (Arrows)
- ▶ Composition

# Composition

- ▶ Morphisms can be composed:
- ▶ If  $f : A \rightarrow B$  and  $g : B \rightarrow C$ , then  $g \circ f : A \rightarrow C$
- ▶ Composition must satisfy some rules

## Composition. Associativity

- ▶ For all  $f : A \rightarrow B$ ,  $g : B \rightarrow C$ ,  $h : C \rightarrow D$ ,

$$h \circ (g \circ f) = (h \circ g) \circ f$$

- ▶ Grouping does not change the result

## Composition. Identity

- ▶ Every object  $A$  has an identity morphism  $\mathbf{id}_A : A \rightarrow A$
- ▶ For any  $f : A \rightarrow B$ ,

$$\mathbf{id}_B \circ f = f = f \circ \mathbf{id}_A$$

- ▶ Identity acts as a "do nothing" operation

# Category

- ▶ A collection of Objects
- ▶ Between any two objects,  $A$  and  $B$  a collection of Morphisms  $Hom(A, B)$
- ▶ For each object  $A$ , an identity morphism  $\mathbf{id}_A : A \rightarrow A$ .
- ▶ For each triple of objects  $A, B, C$ , a **composition law**: If  $f : A \rightarrow B$  and  $g : B \rightarrow C$ , then  $g \circ f : A \rightarrow C$ .

Associativity: For any  $f : A \rightarrow B$ ,  $g : B \rightarrow C$ ,  $h : C \rightarrow D$ ,

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Identity: For any  $f : A \rightarrow B$ ,

$$\mathbf{id}_B \circ f = f = f \circ \mathbf{id}_A.$$

# Isomorphisms

$$f \circ g = \mathbf{id}_A$$

$$g \circ f = \mathbf{id}_B$$

$A$  and  $B$  "are alike" in the category.

# Hom-set

- ▶ For objects  $A, B$  in  $\mathcal{C}$ ,  $\mathbf{Hom}(A, B)$  is the set of all morphisms from  $A$  to  $B$ .
- ▶ Generalizes the idea of "all arrows you can build from  $A$  to  $B$ ".

## Initial Object

- ▶ An object  $0$  such that for every object  $A$ , there is a unique morphism  $0 \rightarrow A$ .
- ▶ In **Set**, the empty set  $\emptyset$ .
- ▶ Encodes the idea of the "least" or "universal starting point".

# Terminal Object

- ▶ An object  $1$  such that for every object  $A$ , there is a unique morphism  $A \rightarrow 1$ .
- ▶ In **Set**, any singleton set.
- ▶ Encodes the idea of the "greatest" or "universal endpoint".

# Product

- ▶ Given objects  $A$  and  $B$ , product  $A \times B$  comes with projections  $A \times B \rightarrow A$  and  $A \times B \rightarrow B$ .
- ▶ For any  $X$  with morphisms  $X \rightarrow A$ ,  $X \rightarrow B$ , there is a unique morphism  $X \rightarrow A \times B$ .
- ▶ In **Set**: Cartesian product of sets.
- ▶ Models "pairing" data.

# CoProduct

- ▶ Given objects  $A$  and  $B$ , coproduct  $A + B$  comes with injections  $A \rightarrow A + B$  and  $B \rightarrow A + B$ .
- ▶ For any  $X$  with morphisms  $A \rightarrow X$ ,  $B \rightarrow X$ , there is a unique morphism  $A + B \rightarrow X$ .
- ▶ In **Set**: disjoint union of sets.
- ▶ Models "either this or that" (sum types).

# Semi-Ring. Algebra of Types

- ▶ Types and their sums/products behave like a semiring:
  - ▶  $0, +$ : initial object and sum (addition)
  - ▶  $1, \times$ : terminal object and product (multiplication)
  - ▶ Distributive law:  $A \times (B + C) \cong (A \times B) + (A \times C)$



$$\text{Maybe}(a) = 1 + a$$



