Category Theory

From objects and arrows to Algebra

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Category Theory

Category

- Objects
- Morphisms (Arrows)
- Composition

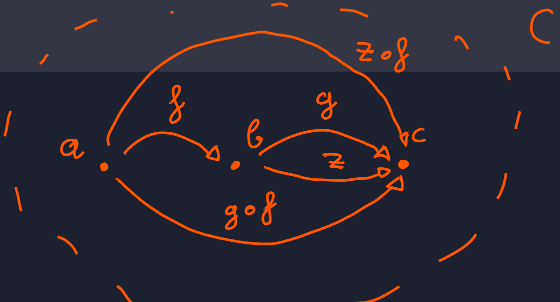
Set

$$\begin{tikzcd} a \arrow[d, bend left] \arrow[d, bend left=60, "f"] \\ b \end{tikzcd} \qquad \cdot\ \emptyset$$

Composition

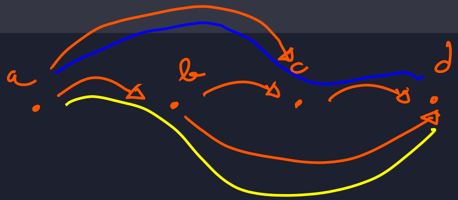
$f: a \to b$

$g: b \to c$



- Morphisms can be composed:
- If $f : A \to B$ and $g : B \to C$, then $g \circ f : A \to C$
- Composition must satisfy some rules

Composition. Associativity

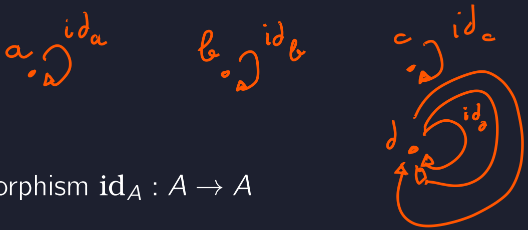


- For all $f : A \to B$, $g : B \to C$, $h : C \to D$,

$$h \circ (g \circ f) = (h \circ g) \circ f$$

- Grouping does not change the result

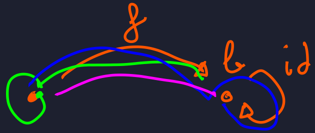
Composition. Identity



- Every object A has an identity morphism $\mathbf{id}_A : A \to A$
- For any $f : A \to B$,

$$\mathbf{id}_B \circ f = f = f \circ \mathbf{id}_A$$

- Identity acts as a "do nothing" operation



$$id_b \circ f = f = f \circ id_a$$

Category

- A collection of Objects
- Between any two objects, A and B a collection of Morphisms $Hom(A, B)$
- For each object A, an identity morphism $\mathbf{id}_A : A \to A$.
- For each triple of objects A, B, C, a **composition law**: If $f : A \to B$ and $g : B \to C$, then $g \circ f : A \to C$. $\circ$

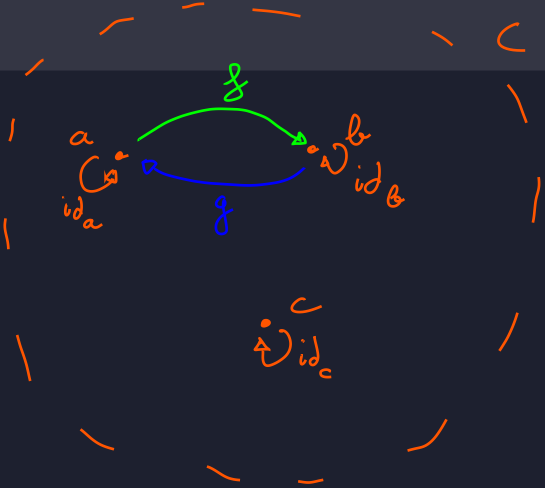
Associativity: For any $f : A \to B$, $g : B \to C$, $h : C \to D$,

$$h \circ (g \circ f) = (h \circ g) \circ f$$

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & b \arrow[r, "g"] & c \end{tikzcd}$$

Identity: For any $f : A \to B$,

$$\mathbf{id}_B \circ f = f = f \circ \mathbf{id}_A.$$

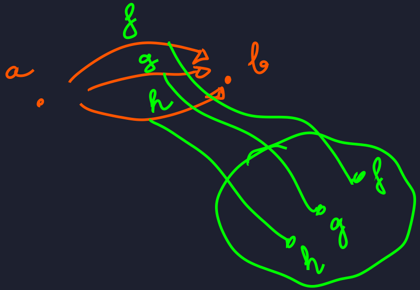


$$f \circ g = \mathrm{id}_A$$
$$g \circ f = \mathrm{id}_B$$

A and B "are alike" in the category.

Hom-set

- For objects A, B in $\mathcal{C}$, $\mathbf{Hom}(A, B)$ is the set of all morphisms from A to B.
- Generalizes the idea of "all arrows you can build from A to B".

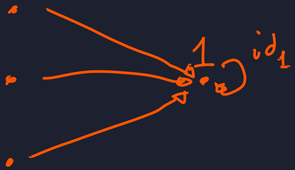


$\mathrm{Hom}(a, b) = \{ f, g, h \}, \dots$

Hom.

- An object 0 such that for every object A, there is a unique morphism $0 \to A$.
- In **Set**, the empty set $\emptyset$. False
- Encodes the idea of the "least" or "universal starting point".

Terminal Object



- An object 1 such that for every object A, there is a unique morphism $A \to 1$.
- In **Set**, any singleton set. True
- Encodes the idea of the "greatest" or "universal endpoint".

Product

Universal Constructions

- Given objects A and B, product $A \times B$ comes with projections $A \times B \to A$ and $A \times B \to B$.
- For any X with morphisms $X \to A$, $X \to B$, there is a unique morphism $X \to A \times B$.
- In **Set**: Cartesian product of sets.
- Models "pairing" data.

$$\begin{tikzcd}
 & p' \arrow[dl, bend right, "q'_a"'] \arrow[d, dashed, "m"] \arrow[dr, bend left, "q'_b"] & \\
a & p = a \times b \arrow[l, "q_a"] \arrow[r, "q_b"'] & b
\end{tikzcd}$$

$p = a \times b$

$(a, b, c) \xrightarrow{m} (a, b)$

q_a : first( )

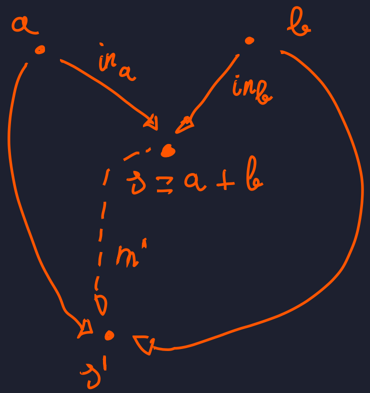
q_b : snd ()

m = discard third

CoProduct

- Given objects A and B, coproduct $A + B$ comes with injections $A \to A + B$ and $B \to A + B$.
- For any X with morphisms $A \to X$, $B \to X$, there is a unique morphism $A + B \to X$.
- In **Set**: disjoint union of sets.
- Models "either this or that" (sum types).

enum
Either a b



$in_a : a \longrightarrow z$ Left(a)
$in_b : b \longrightarrow z$ Right(b)

Semi-Ring. Algebra of Types

- Types and their sums/products behave like a semiring:
  - 0, $+$: initial object and sum (addition)
  - 1, $\times$: terminal object and product (multiplication)
  - Distributive law: $A \times (B + C) \cong (A \times B) + (A \times C)$

$$bool = 2 = 1 + 1$$

Unit

enum ~~Either~~ ( ) ( )

Left() Right()

result = match two {
  true => 2+3
  false => 7+8
}

<= Rename

if [ ]
else [ ]

result = match two {
  Left(_) => { } // 2+3
  Right(_) => { } // 7+8
}

Maybe/Option

$$Maybe(a) = 1 + a$$

```
enum Option {
  Some(T)    <- a
  None
}
```

List Equation

$$L(a) = 1 + a \times L(a)$$

$$= 1 + a \times (1 + a\, L(a))$$

$$= 1 + a + aa + aaa + \ldots$$

$$(a \times b + c) \times d = a \times b \times d + c \times d$$

$$\underbrace{(\text{Either}(a, b)\ c,\ d)}_{1} = \underbrace{\text{Either}\ (a, b, d)\ (c, d)}_{2}$$

$$f(1) \rightleftarrows f(2)$$