ODEs

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ODE Zoology

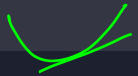
ODE Analytical Methods

ODEs. Numerical Methods

ODE Zoology

- Order
- Linearity
- Homogeneity
- Autonomy
- Constant/Variable Coefficients
- Exactness
- Separability

ODE Zoology. Order



$y' = \frac{dy}{dx} = f(x)$ $\quad$ $y'' = \frac{d^2y}{dx^2}$

- **Definition:** The order of an ordinary differential equation (ODE) is the index of the highest derivative of the unknown function.
- **First-Order ODE:**
  - Example: $y' = f(x, y)$.
- **Second-Order ODE:**
  - Example: $y'' + p(x)y' + q(x)y = g(x)$.
- Higher-order ODEs can be reduced to systems of first-order ODEs.

Converting nth Order ODE to System of First-Order ODEs

- **General nth Order ODE:**
  - $y^{(n)} = F(x, y, y', y'', \ldots, y^{(n-1)})$
  - Introduce new variables:

$$y_1 = y, \quad y_2 = y', \ldots, y_n = y^{(n-1)}$$

  - The system of first-order ODEs:

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= y_3, \\ &\vdots \\ y_{n-1}' &= y_n, \\ y_n' &= F(x, y_1, y_2, \ldots, y_n) \end{aligned}$$

ODE Zoology. Linearity

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad (1)$$

where $a_i(x)$ and $g(x)$ are given functions.

- **Definition:** A differential equation is called homogeneous if all its terms are a function of the dependent variable and its derivatives only, and it equals zero.
- **Linear Homogeneous ODE:**
  - Example: $y'' + p(x)y' + q(x)y = 0$.
- **Non-linear Homogeneous ODE:**
  - Example: $y'' + y^3 = 0$.
- **Properties:**
  - Solutions can be scaled: if $y(x)$ is a solution, then $c \cdot y(x)$ is also a solution for any constant c.

ODE Zoology. Linear + Homogeneous

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (2)$$

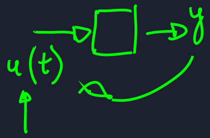
- **Superposition Principle:** If $y_1(x)$ and $y_2(x)$ are solutions, then $c_1 y_1(x) + c_2 y_2(x)$ is also a solution for any constants c_1 and c_2.
- **Existence of a Basis:** The set of all solutions forms a vector space, with a basis consisting of linearly independent solutions.
- **Wronskian** $\approx$ Determinant of basis vectors.
  If $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for some x, then these solutions are linearly independent.

ODE Zoology. Autonomy

- **Definition:** An ODE is autonomous if it does not explicitly depend on the independent variable.
- **Autonomous Form:**
  - $\frac{dy}{dx} = f(y)$ (first-order)
  - $\frac{d^2y}{dx^2} = f(y, y')$ (second-order)
- **Non-Autonomous Form:**
  - $\frac{dy}{dx} = f(x, y)$ (explicitly depends on x)
- Time-translation invariance: if $y(x)$ is a solution, then $y(x + c)$ is also a solution

ODE Zoology. Autonomy on Dynamical Systems

- **Autonomous ODEs:** Can "advance by themselves"
  - Self-contained system, no external driving function needed
- **Non-Autonomous ODEs:** Require external information
  - Need explicit form/mapping of how system depends on x
  - Externally driven or forced systems



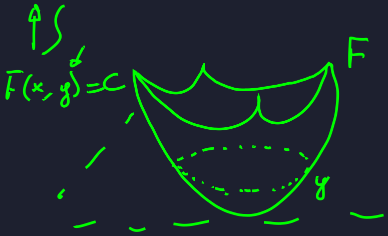
ODE Zoology. Constness of Coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = g(x) \tag{3}$$

$a_i \in \mathbb{R}$

- **Exact Differential Equation:**
  - Form: $M(x,y)dx + N(x,y)dy = 0$
  - **Exactness condition:** $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
  - Represents total differential: $dF(x,y) = 0$ for some function F

$\Leftrightarrow \quad M(x,y) + N(x,y)\frac{dy}{dx} = 0$



ODEs. Analytical Methods

- Separation of Variables
- Characteristic Equation
- Variation of Parameters
- and more...

Separation of Variables

- **Applies to:** First-order ODEs of the form $\frac{dy}{dx} = g(x)h(y)$
- **Method:**
  1. Rewrite as $\frac{dy}{h(y)} = g(x)dx$
  2. Integrate both sides: $\int \frac{dy}{h(y)} = \int g(x)dx$
- **Example:** $\frac{dy}{dx} = xy$
  - $\int \frac{dy}{y} = \int xdx$
  - $\ln|y| = \frac{x^2}{2} + C$
  - $y = Ae^{x^2/2}$

Characteristic Equation Method

- **General form:** $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$
- **Characteristic Eq:** $a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0$
- **General Solution cases:**
  - **Distinct Roots** $(z_1 \neq z_2 \neq \ldots \neq z_n)$:

$$y(x) = C_1 e^{z_1 x} + C_2 e^{z_2 x} + \ldots + C_n e^{z_n x}$$

  - **Complex Roots** $(z_j = \alpha + i\beta)$:

$$y_j(x) = e^{\alpha x} \cdot (C_1 \cos(\beta x) + C_2 \sin(\beta x))$$

  - **Repeated Roots** $(z_j,\ \text{multiplicity} = m_j)$:

$$y_j(x) = (C_1 + C_2 x + \ldots + C_{m_j} x^{m_j - 1}) e^{z_j x}$$

- **General Solution:**

$$y(x) = \sum_{j=1}^{k} \left(C_{j1} + C_{j2} x + \ldots + C_{j m_j} x^{m_j - 1} \right) e^{z_j x}$$

where m_j is the multiplicity of the j-th root

Characteristic Equation Method

- **General Solution:**

$$y(x) = \sum_{j=1}^{k} \left(C_{j1} + C_{j2}x + \ldots + C_{jm_j}x^{m_j-1} \right) e^{z_j x}$$

where m_j is the multiplicity of the j-th root

The form of solutions to Linear Non-Homogeneous ODEs

$L[y] = 0 \sim Ax = 0$

$$L[y] = f(x)$$

where L is a linear differential operator

- **General Solution Structure:**

$$y = y_h + y_p$$

Affine

  - y_h: General solution to homogeneous equation $L[y] = 0$
  - y_p: Any particular solution to $L[y] = f(x)$
- **Why:** If y_1, y_2 are solutions to $L[y] = f(x)$, then:

$$L[y_1 - y_2] = L[y_1] - L[y_2] = f(x) - f(x) = 0$$

So $(y_1 - y_2)$ satisfies the homogeneous equation
So $y_1 = y_h + y_2$

Variation of Parameters

- **General Higher Order Linear ODE:**

NonConst NonHom Linear

$$y^{(n)}(x) + \sum_{i=0}^{n-1} a_i(x) y^{(i)}(x) = b(x)$$

- **Homogeneous Solutions:** $y_1(x), y_2(x), \ldots, y_n(x)$
- **Particular Solution:** $y_p(x) = \sum_{i=1}^{n} c_i(x) y_i(x)$
- **Extra Conditions:** $\sum_{i=1}^{n} c_i'(x) y_i^{(j)}(x) = 0$ for $j = 0, \ldots, n-2$
- **System (Cramer's Rule):**

$$\begin{bmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \\ \vdots \\ c_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b(x) \end{bmatrix}$$

- **Solution:** $c_i' = \frac{W_i(x)}{W(x)}$ where W is the Wronskian

Variation of Parameters

- **Linear System:**

$$\begin{bmatrix} y_{1H} & \cdots & y_{nH} \\ y'_{1H} & \cdots & y'_{nH} \\ \vdots & \ddots & \vdots \\ y_{1H}^{(n-1)} & \cdots & y_{nH}^{(n-1)} \end{bmatrix} \begin{bmatrix} c'_1 \\ c'_2 \\ \vdots \\ c'_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b(x) \end{bmatrix}$$

- **Cramer's Rule Solution:** $c'_i = \frac{W_i(x)}{W(x)}$ for $i = 1, \ldots, n$ where
  - W: Wronskian determinant of the basis $y_{1H} \ldots y_{nH}$
  - W_i: Wronskian det. with i-th column as $[0, 0, \ldots, 0, b(x)]^T$
- **Particular Solution:** $y_p(x) = \sum_{i=1}^{n} y_{iH}(x) \int c'_i(x) dx$
- **Complete Solution:** $y = y_H + y_p$

Euler

$a \in \mathbb{R}$

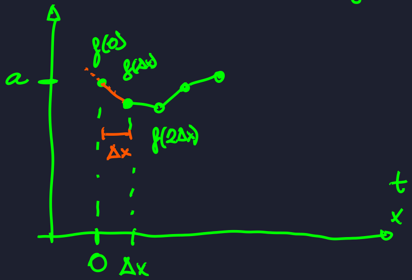
$$\frac{dy}{dx} = f(x)$$

IC: $y(0) = a$

$f(0) = -1$

Δx: $y(\Delta x) = y(0) + f(0) \cdot \Delta x$

haha computer go brrr



RK

$y(t)$