Manifolds

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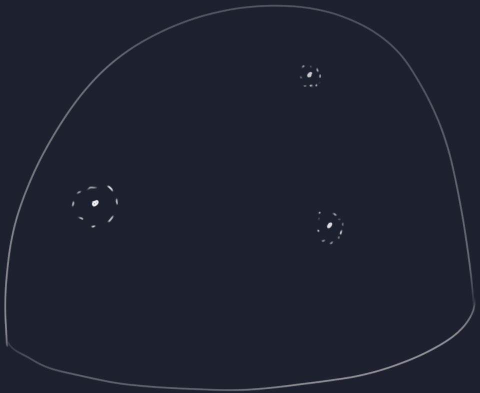
Manifolds

A k-dimensional manifold, M is any topological space that given any point in the manifold its neighborhood looks like a ball in $\mathbb{R}^k$



Topological Space

A topological space is a collection of subsets of X such that:

- $\emptyset$ and X are open.
- Arbitrary unions of open sets are open.
- Finite intersections of open sets are open.

Maifold

A Manifold is a topological space M that is:

- Hausdorff: For any two distinct points, there exist disjoint open sets containing each.
- Second-countable: There exists a countable basis for the topology.
- Locally Euclidean: Every point $x \in M$ has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

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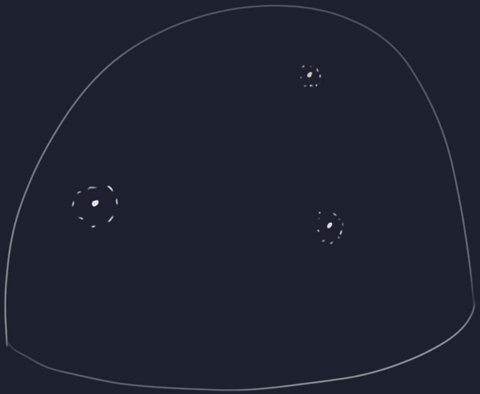

Implicit Representation

M is a k-dimensional manifold if $\forall p \in M$, $\exists U$ such that $p \in U$, and $(n-k)$ differentiable functions $f_1, f_2, \ldots f_{n-k}$ such that

1. U is an open set
2. $M \cap U = (p_1 = 0) \cap \ldots \cap (p_{n-k} = 0)$
3. $\forall p \in M \cap U$ the gradients ∇f_i are linearly independent $\forall i$

The implicit representation of the circle

$$S^1 = \{(x, y) : f = x^2 + y^2 - 1 = 0\}$$

The gradient is always linearly independent

$$\nabla f = (\frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}) = (2x, 2y)$$

Parametric Representation

M is a k-dimensional manifold if $\forall p \in M$ there is an open set U in $\mathbb{R}^k$ containing p and a parametrizing map ϕ such that

$$\phi(\text{Ball in } \mathbb{R}^k) = M \cap U$$

The parametric representation of the circle

$$\varphi : (0, 2\pi) \to S^1 \subset \mathbb{R}^2$$

$$\varphi(\theta) = (\cos\theta, \sin\theta)$$

$$J_\varphi(\theta) = \begin{bmatrix} \frac{\partial}{\partial\theta}\cos\theta \\ \frac{\partial}{\partial\theta}\sin\theta \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Abstract Manifolds

M is an n-dimensional manifold if there is an open cover, where each U_α has an associated homeomorphism:

$$\phi_\alpha : U \to V$$

where $V \subset \mathbb{R}^n$

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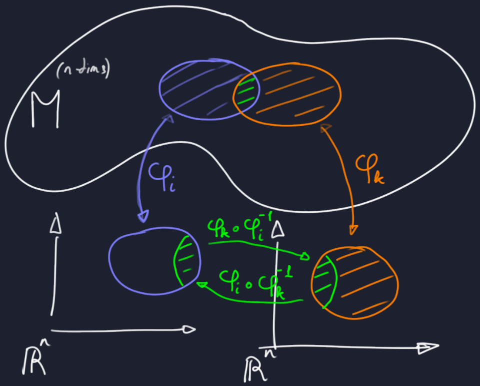
Homeomorphism, a function that is:

- continuous
- bijective
- with a continuous inverse

An homeomorphism is a 1-1 mapping between topological spaces

Abstract Manifolds

- The ϕ_α mappings are called **charts**
- A set of charts $\phi_i : U_\alpha \to V | i \in \mathbb{N}$ is called the **atlas** of M



This mapping allow us to do calculus on $\mathbb{R}^n$, where it is well defined

Stereographic Projection as an Atlas

Define two open sets covering S^1:

- $U_+ = S^1 \setminus \{(0,1)\}$
- $U_- = S^1 \setminus \{(0,-1)\}$

Charts:

$$\varphi_+ : U_+ \to \mathbb{R}$$
$$(x,y) \mapsto \frac{x}{(1-y)}$$
$$\varphi_- : U_- \to \mathbb{R}$$
$$(x,y) \mapsto \frac{x}{(1+y)}$$

Atlas:

$$(\{U_+, U_-\}, \{\varphi_+, \varphi_-\})$$

Stereographic Projection as an Atlas