Curves, Curvature and Torsion

20250818 CMK Study Group

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August 18, 2025

- Parametrized Curves
- Principal Curvature
- Symbolics.jl
- Torsion

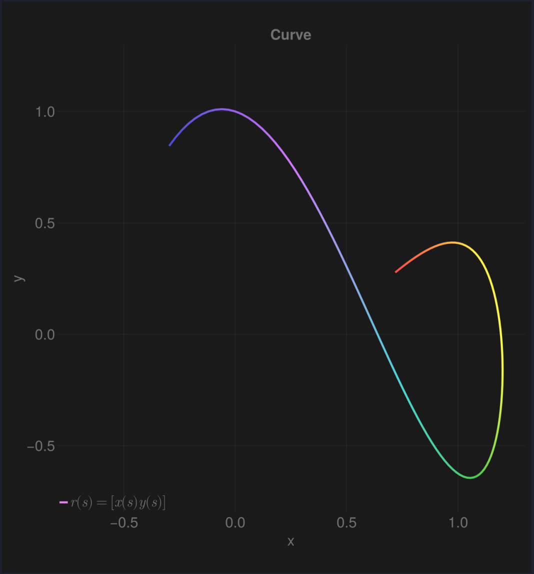
Parametrized Curves

$$s \in [-0.1, 1]$$
$$x(s) = 1.2\sin(2.5s)$$
$$y(s) = \cos(7s)\,e^{-s}$$
$$\mathbf{r} = \begin{bmatrix} x(s) \\ y(s) \end{bmatrix}$$

The curve is not a function, but it is a Manifold in $\mathbb{R}^2$

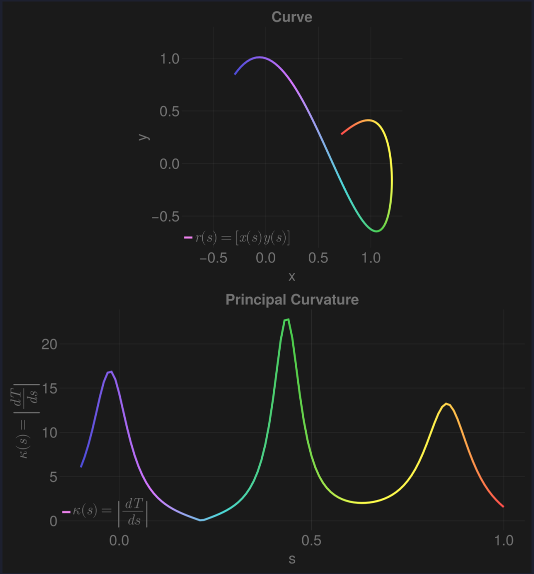
The curve r is parametrized by s

In the figure, the value of s changes with the color



Principal Curvature

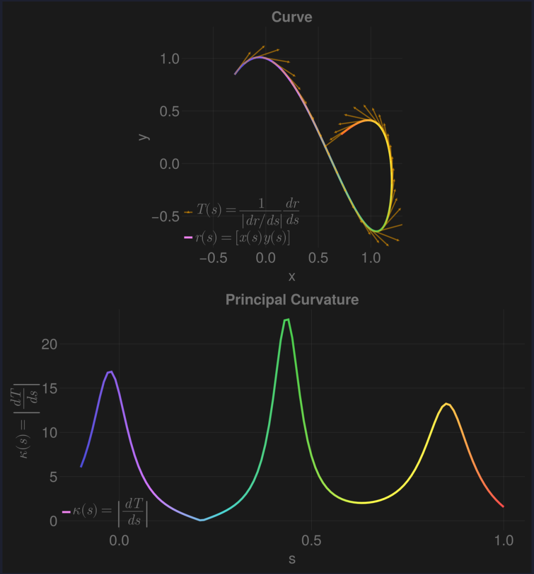
The principal curvature provides a metric for how much the curve is changing direction inside the plane



Principal Curvature

The principal curvature is computed by the rate of change of the normalized tangent

$$T(s) = \frac{\frac{dr}{ds}}{\left\|\frac{dr}{ds}\right\|} \tag{1}$$

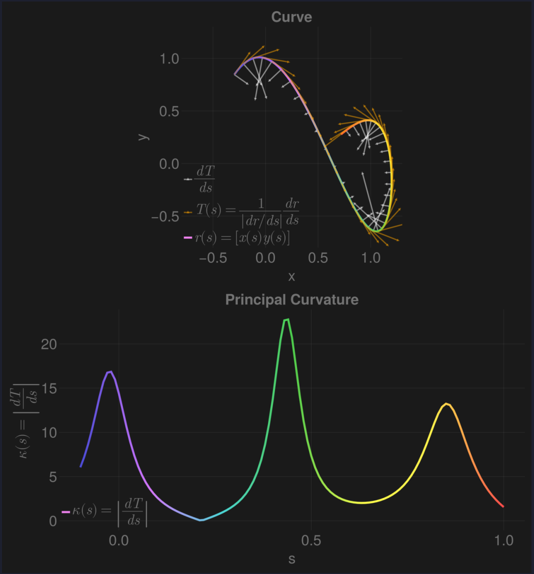


Principal Curvature

The **principal curvature** is the norm of the rate of change of the arc-length tangent.

$$\kappa(s) = \left| \frac{dT}{ds} \right| \quad (2)$$

or the norm of the normal vector.



Symbolics.jl

Symbolics.jl is a Computer Algebra System (CAS) for Julia.
Consider:

$$x(s) = sin(2.5s) * 1.2$$

$$y(s) = cos(7s) * \exp(-s)$$

$$\mathbf{r} = \begin{bmatrix} x(s) \\ y(s) \end{bmatrix}$$

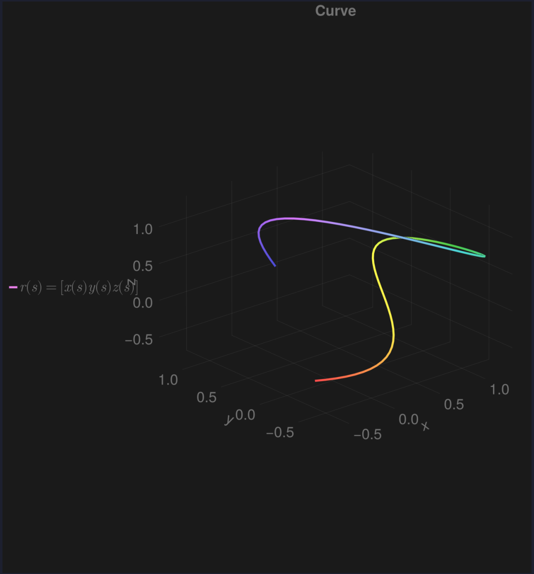
Say we want to calculate $\frac{dT}{ds}$ where $T = \frac{\frac{dr}{ds}}{||\frac{dr}{ds}||}$

Symbolics.jl Differentiation

```
julia> @variables s
1-element Vector{Num}:
 s

julia> x(s) = sin(2.5 * s) * 1.2
x (generic function with 1 method)

julia> y(s) = cos(7 * s) * exp(-1 * s)
y (generic function with 1 method)

julia> r = [x(s) y(s)]
1×2 Matrix{Num}:
 1.2sin(2.5s)  exp(-s)*cos(7s)

julia> drds = Symbolics.derivative(r, s)
1×2 Matrix{Num}:
 3.0cos(2.5s)  -exp(-s)*cos(7s) - 7exp(-s)*sin(7s)

julia> tangent_norm = drds / sqrt(sum(abs2, drds))
1×2 Matrix{Num}:
 (3.0cos(2.5s)) / sqrt(abs2(3.0cos(2.5s)) + abs2(-exp(-s)*cos(7s) - 7exp(-s)*sin(7s)))  …  (-exp(-s)*cos(7s) - 7exp(-s)*sin(7s)) / sqrt(abs2(3.0cos(2.5s)
) + abs2(-exp(-s)*cos(7s) - 7exp(-s)*sin(7s)))

julia> dTds = Symbolics.derivative(tangent_norm, s)
1×2 Matrix{Num}:
 (-7.5sin(2.5s)) / sqrt(abs2(3.0cos(2.5s)) + abs2(-exp(-s)*cos(7s) - 7exp(-s)*sin(7s))) + (-((3.0cos(2.5s)) / (sqrt(abs2(3.0cos(2.5s)) + abs2(-exp(-s)*co
s(7s) - 7exp(-s)*sin(7s)))^2))*(-45.0sin(2.5s)*cos(2.5s) + (-2exp(-s)*cos(7s) - 14exp(-s)*sin(7s))*(-48exp(-s)*cos(7s) + 14exp(-s)*sin(7s)))) / (2sqrt(ab
s2(3.0cos(2.5s)) + abs2(-exp(-s)*cos(7s) - 7exp(-s)*sin(7s))))  …  (-48exp(-s)*cos(7s) + 14exp(-s)*sin(7s)) / sqrt(abs2(3.0cos(2.5s)) + abs2(-exp(-s)*cos
(7s) - 7exp(-s)*sin(7s))) + (-(-45.0sin(2.5s)*cos(2.5s) + (-2exp(-s)*cos(7s) - 14exp(-s)*sin(7s))*(-48exp(-s)*cos(7s) + 14exp(-s)*sin(7s)))*((-exp(-s)*co
s(7s) - 7exp(-s)*sin(7s)) / (sqrt(abs2(3.0cos(2.5s)) + abs2(-exp(-s)*cos(7s) - 7exp(-s)*sin(7s)))^2))) / (2sqrt(abs2(3.0cos(2.5s)) + abs2(-exp(-s)*cos(7s
) - 7exp(-s)*sin(7s))))
```

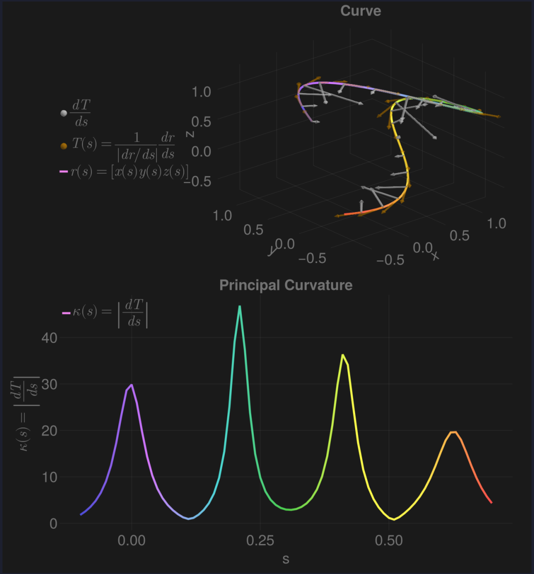
Parametrized Curves in $\mathbb{R}^3$

$$s \in [-0.1, 0.7]$$
$$x(s) = \sin(5s) \cdot 1.2$$
$$y(s) = 0.8\cos(15s) \cdot e^{-0.5s}$$
$$z(s) = \cos(3s)$$
$$\mathbf{r} = \begin{bmatrix} x(s) \\ y(s) \\ z(s) \end{bmatrix}$$



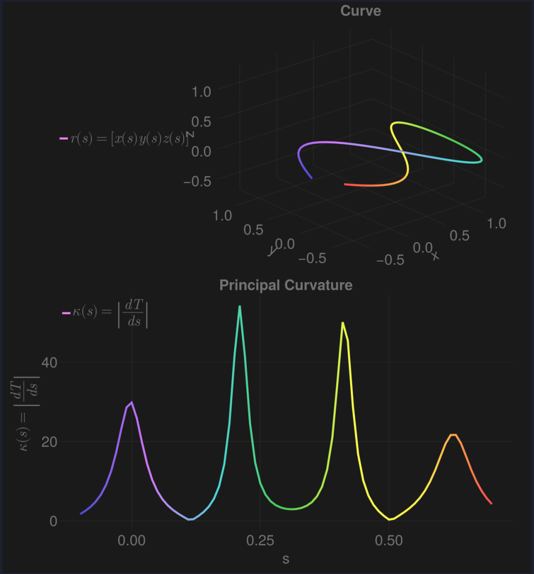
Principal Curvature in $\mathbb{R}^3$

All works just the same



Principal Curvature in $\mathbb{R}^3$

But… There has to be more



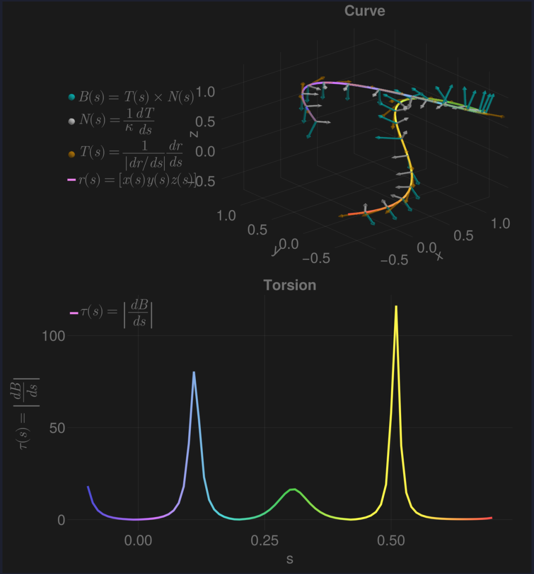
Torsion

$$N(s) = \frac{\frac{dT}{ds}}{\left\|\frac{dT}{ds}\right\|} = \frac{1}{\kappa}\frac{dT}{ds}$$

$$B(s) = T(s) \times N(s)$$

$N(s)$ is the normalized **normal vector**, also called the **principal normal**.

$B(s)$ is the **binormal vector**



Torsion

The **torsion** is the rate of change of the binormal vector

$$\tau(s) = \left\| \frac{dB}{ds} \right\| \tag{3}$$