20250721 Cool Math Kids CMK Group

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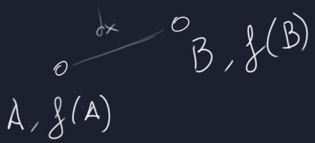
Stokes Theorem

Stokes Theorem References

- All the math you missed. Chapters 5 and 6
- Understanding Vector Calculus by Gabriele Carcassi
- Calculus Wikipedia Series

Gradient

The gradient provides information about the rate of change of a function



$$\Delta f = f(B) - f(A)$$

$$\text{grad}_x(U) = \lim_{dx \to 0} \frac{f(x+dx) - f(x)}{dx} = \frac{\partial f}{\partial x}$$

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots \frac{\partial f}{\partial x_n} \right)$$

Nabla Operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right) \tag{1}$$

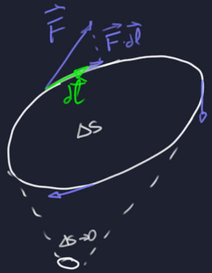
Gradient (Fundamental Theorem of Calculus and Gradient Theorem)

$$\int_a^b \frac{dF}{dx} dx = f(b) - f(a) \tag{2}$$

$$\int_L \nabla \vec{F} d\vec{l} = f(L_{end}) - f(L_{start}) \tag{3}$$

The **curl** provides local information about how much a vector field **"rotates" around a point**

$$\overrightarrow{curl}(\vec{F}) = \lim_{\Delta S \to 0} \frac{\oint_l \vec{F}\vec{dl}}{\Delta S} \tag{4}$$



Curl

$$\overrightarrow{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \quad (5)$$

Curl (Curl Theorem, Kelvin-Stokes Theorem)

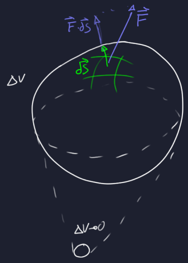
$$\iint_S (\nabla \times \vec{F}) d\vec{S} = \oint_{\delta S} \vec{F} d\vec{l} \tag{6}$$

"The total whirliness on the surface" = "The total flow through the boundary"

Divergence

The **divergence** provides local information about how much a vector field is **"spreading" out at a point**

$$div(\vec{F}) = \lim_{\Delta V \to 0} \frac{\oiint \vec{F} \cdot d\vec{S}}{\Delta V} \quad (7)$$



Divergence

$$div(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x^1} + \frac{\partial F_2}{\partial x^2} + \cdots + \frac{\partial F_n}{\partial x^n} \tag{8}$$

Divergence (Divergence Theorem, Greens Theorem)

$$\iiint_V (\nabla \cdot \vec{F}) d\vec{V} = \oiint_{\delta V} \vec{F} \cdot d\vec{S} \tag{9}$$

"The total spreading out in V = The total flow across the boundary S"

$$\int_L \nabla \vec{F} d\vec{l} = f(L_{end}) - f(L_{start})$$

$$\iint_S (\nabla \times \vec{F}) d\vec{S} = \oint_{\delta S} \vec{F} d\vec{l}$$

$$\iiint_V (\nabla \cdot \vec{F}) d\vec{V} = \oiint_{\delta V} \vec{F} \cdot d\vec{S}$$

Integral Derivative on Object = Integral Field on Boundary

Stokes Theorem

$$\int_M d\omega = \int_{\delta M} \omega \tag{10}$$

- M: k-dimensional manifold in $\mathbb{R}^n$
- δM: Boundary of M
- ω: Differential (k-1)-form
- $d\omega$: Exterior derivative of ω

"Integral of the derivative on the interior = Integral on the boundary"

Manifolds

Definition

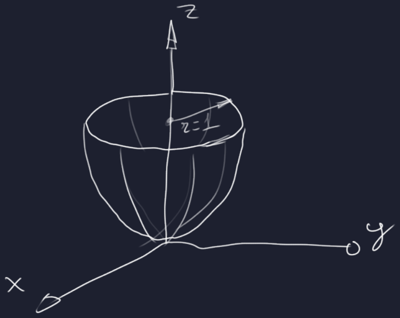
A *differentiable manifold* M of dimension k in $\mathbb{R}^n$ is a set of points in $\mathbb{R}^n$ such that for any point $p \in M$, there is a small open neighborhood U of p, a vector-valued differentiable function $F : \mathbb{R}^k \to \mathbb{R}^n$ and an open set V in in $\mathbb{R}^k$ with

- $F(V) = U \cap M$
- The Jacobian of F has rank k at every point in V

The function F is called the **parametrization** of the manifold.

Manifold Example

$$F : \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \to \mathbb{R}^3$$
$$F(x, y) = (x, y, x^2 + y^2)$$



k-forms

A k-form is a mapping in which given a Manifold, you provide a point and k vectors from the tangent space of the manifold and a Scalar is returned.

- 0-form $\rightarrow f(x)$. A function. You provide a point, the function value is returned.
- 1-form $\rightarrow \omega = a(x)\, dx$. You provide a tangent vector at a point, the 1-form returns a number (linear map on tangent vectors).
- 2-form $\rightarrow \omega = b(x)\, dx \wedge dy$. You provide two tangent vectors at a point, the 2-form returns a number (alternating bilinear map).

0-forms

A 0-form is a differentiable function on a manifold.

$$F : M \to \mathbb{R}$$

Elementary 1-forms

They form the **basis** for the **vector space** of 1-forms

3 Elemenentary 1-forms in $\mathbb{R}^3$

- dx
- dy
- dz

All other 1-forms are linear combinations of these basis elements

Elementary 2-forms

They form the **basis** for the **vector space** of 2-forms

3 Elemenentary 2-forms in $\mathbb{R}^3$

- $dy \wedge dz$
- $dx \wedge dz$
- $dx \wedge dy$

All other 2-forms are linear combinations of these basis elements

k-forms

$$\omega : p \mapsto (\omega_p : T_pM \to \mathbb{R}) \tag{11}$$

Provided a point (p), and a vector from the tangent space of the Manifold (M) at that point (T_pM) a scalar is returned.
"Provide a point, and a function is returned, that function given a tangent vector to the manifold returns a scalar"

k-forms

For a given $\mathbb{R}^n$ space there are $\binom{n}{k}$ elementary k-forms.
Each k-form can be written as:

$$\omega = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} f_{i_1 \ldots i_k}(x)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} \tag{12}$$

The exterior derivative

$$d : \text{k-forms} \rightarrow \text{(k+1)-forms} \tag{13}$$

Given $\omega = \sum_{\forall I} f_I dx_I$ (a k-form)

$$d\omega = \sum_{\forall I} df_I \wedge dx_I \tag{14}$$

The exterior derivative of 0-forms

Assume $\mathbb{R}^3$
$d(\text{0-form}) \rightarrow \text{1-form}$
$\omega_0 = f(x_1, x_2, x_2)$

$$df = \frac{\delta f}{\delta dx_1} dx_1 + \frac{\delta f}{\delta dx_2} dx_2 + \frac{\delta f}{\delta dx_3} dx_3 \tag{15}$$

The exterior derivative of 1-forms

Assume $\mathbb{R}^3$

$d(\text{1-form}) \rightarrow \text{2-form}$

$\omega_1 = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$

$$\begin{aligned} d\omega &= df_1 \wedge dx_1 + df_2 \wedge dx_2 + df_3 \wedge dx_3 \\ &= (\frac{\delta f_1}{\delta x_1} dx_1 + \frac{\delta f_1}{\delta x_2} dx_2 + \frac{\delta f_1}{\delta x_3} dx_3) \wedge dx_1 + (..) \wedge dx_2 + (..) \wedge dx_3 \\ &= (\frac{\delta f_3}{\delta x_2} - \frac{\delta f_2}{\delta x_3}) dx_2 \wedge dx_3 + (\frac{\delta f_1}{\delta x_3} - \frac{\delta f_3}{\delta x_1}) dx_3 \wedge dx_1 + (\frac{\delta f_2}{\delta x_1} - \frac{\delta f_1}{\delta x_2}) dx_1 \wedge dx_2 \end{aligned}$$

The exterior derivative of 1-forms

Assume $\mathbb{R}^3$
d(1-form) $\rightarrow$ 2-form
$\omega_1 = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$

$$\begin{aligned} d\omega &= df_1 \wedge dx_1 + df_2 \wedge dx_2 + df_3 \wedge dx_3 \\ &= (\frac{\delta f_1}{\delta x_1} dx_1 + \frac{\delta f_1}{\delta x_2} dx_2 + \frac{\delta f_1}{\delta x_3} dx_3) \wedge dx_1 + (..) \wedge dx_2 + (..) \wedge dx_3 \\ &= (\frac{\delta f_3}{\delta x_2} - \frac{\delta f_2}{\delta x_3}) dx_2 \wedge dx_3 + (\frac{\delta f_1}{\delta x_3} - \frac{\delta f_3}{\delta x_1}) dx_3 \wedge dx_1 + (\frac{\delta f_2}{\delta x_1} - \frac{\delta f_1}{\delta x_2}) dx_1 \wedge dx_2 \end{aligned}$$

Which is precisely the curl:

$$\nabla \times \vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$