20250721 Cool Math Kids CMK Group

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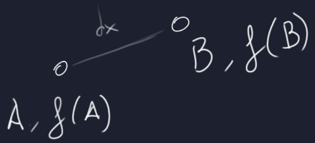
Stokes Theorem

Stokes Theorem References

- All the math you missed. Chapters 5 and 6
- Understanding Vector Calculus by Gabriele Carcassi
- Calculus Wikipedia Series

Gradient

The gradient provides information about the rate of change of a function



$$\Delta f = f(B) - f(A)$$

$$\mathrm{grad}_x(U) = \lim_{dx \to 0} \frac{f(x+dx) - f(x)}{dx} = \frac{\partial f}{\partial x}$$

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots \frac{\partial f}{\partial x_n} \right)$$

Nabla Operator

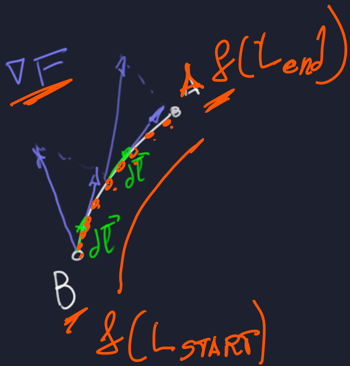
$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right) \tag{1}$$

$$\int_a^b \frac{dF}{dx}\,dx = f(b) - f(a) \tag{2}$$

$$\int_L \nabla \vec{F}\, d\vec{l} = f(\overset{0\ \ A}{L_{end}}) - f(\overset{0\ \ B}{L_{start}}) \tag{3}$$

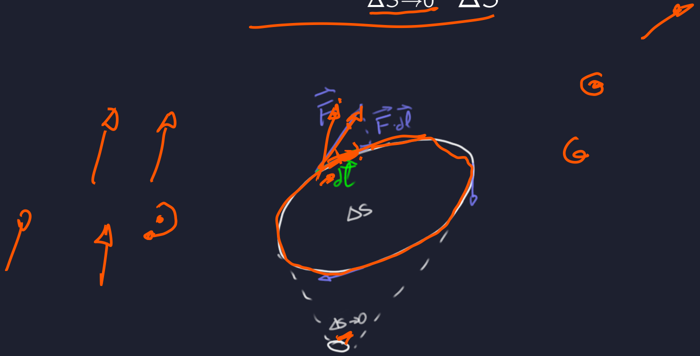
$dx \to 0$

Σ



The **curl** provides local information about how much a vector field **"rotates" around a point**

$$\overrightarrow{curl}(\vec{F}) = \lim_{\Delta S \to 0} \frac{\oint_l \vec{F} d\vec{l}}{\Delta S} \tag{4}$$



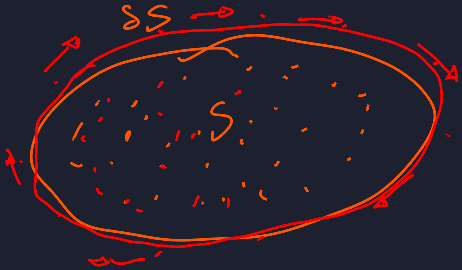
$F: \mathbb{R}^3 \to \mathbb{R}^3$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} 0.2 \\ 0.3 \\ 0 \end{matrix} \qquad \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \begin{matrix} 7 \\ -4 \\ 3 \end{matrix}$

$$\overrightarrow{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \underbrace{\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)}_{curl \in \mathbb{R}^3} \tag{5}$$

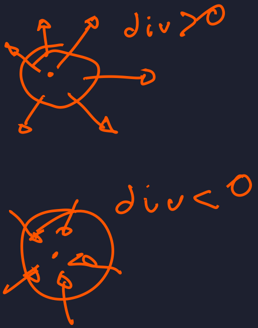
$$\iint_S (\underbrace{\overset{\text{curl}}{\nabla \times \vec{F}}}_{})d\vec{S} = \oint_{\delta S} \vec{F} d\vec{l} \tag{6}$$

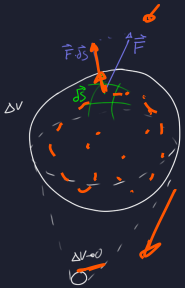
"The total whirliness on the surface" = "The total flow through the boundary"



The **divergence** provides local information about how much a vector field is **"spreading" out at a point**

$$div(\vec{F}) = \lim_{\Delta V \to 0} \frac{\oiint \vec{F} \cdot d\vec{S}}{\Delta V} \tag{7}$$





Divergence

$$div(\vec{F}) = \underbrace{\nabla \cdot \vec{F}} = \frac{\partial F_1}{\partial x^1} + \frac{\partial F_2}{\partial x^2} + \cdots + \frac{\partial F_n}{\partial x^n} \quad (8)$$

∇F : Grad — Prod

$\nabla \times F$: Curl — Vector Prod

$\nabla \cdot F$: Div — Inner Prod

Divergence (Divergence Theorem, Greens Theorem)

$$\iiint_V (\nabla \cdot \vec{F}) d\vec{V} = \oiint_{\delta V} \vec{F} \cdot d\vec{S} \tag{9}$$

"The total spreading out in V = The total flow across the boundary S"

+1D

+1D

$$\int_L \nabla \vec{F} d\vec{l} = f(L_{end}) - f(L_{start})$$

$$\iint_S (\nabla \times \vec{F}) d\vec{S} = \oint_{\delta S} \vec{F} d\vec{l}$$

$$\iiint_V (\nabla \cdot \vec{F}) d\vec{V} = \oiint_{\delta V} \vec{F} \cdot d\vec{S}$$

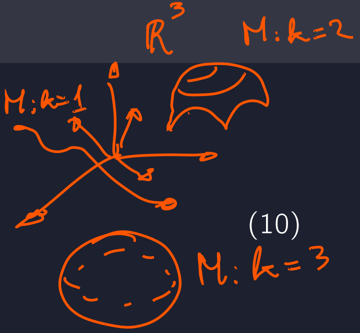
Integral Derivative on Object = Integral Field on Boundary

Stokes Theorem

$k \leq n$

$$\int_M d\omega = \int_{\delta M} \omega \qquad (10)$$

∂M

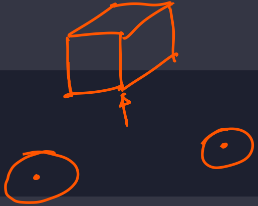


- M: k-dimensional manifold in $\mathbb{R}^n$
- δM: Boundary of M
- ω: Differential (k-1)-form
- $d\omega$: Exterior derivative of ω

k-form

"Integral of the derivative on the interior = Integral on the boundary"

Manifolds



Definition

A *differentiable manifold* M of dimension k in $\mathbb{R}^n$ is a set of points in $\mathbb{R}^n$ such that for any point $p \in M$, there is a small open neighborhood U of p, a vector-valued differentiable function $F : \mathbb{R}^k \to \mathbb{R}^n$ and an open set V in in $\mathbb{R}^k$ with

- $F(V) = U \cap M$
- The Jacobian of F has rank k at every point in V

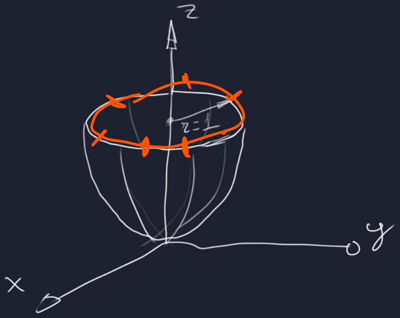
The function F is called the **parametrization** of the manifold.

Manifold Example

$$F : \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \to \mathbb{R}^3$$

$$F(x, y) = (x, y, x^2 + y^2)$$

$\mathbb{R}$



k-forms

A k-form is a mapping in which given a Manifold, you provide a point and k vectors from the tangent space of the manifold and a Scalar is returned.

- 0-form $\rightarrow f(x)$. A function. You provide a point, the function value is returned.
- 1-form $\rightarrow \omega = a(x)\,dx$. You provide a tangent vector at a point, the 1-form returns a number (linear map on tangent vectors).
- 2-form $\rightarrow \omega = b(x)\,dx \wedge dy$. You provide two tangent vectors at a point, the 2-form returns a number (alternating bilinear map).

0-forms

A 0-form is a differentiable function on a manifold.

$$F : M \to \mathbb{R}$$

Vector Spaces

They form the **basis** for the **vector space** of 1-forms

3 Elemenentary 1-forms in $\mathbb{R}^3$

- dx
- dy
- dz

$$\omega_1 = \sum_i \alpha_i dx_i$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} : \mathbb{R}^3$$

$$\left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\; e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\; e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right. \qquad \sum \alpha_i e_i$$

All other 1-forms are linear combinations of these basis elements

Elementary 2-forms

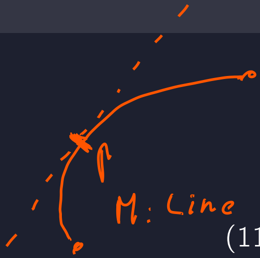
They form the **basis** for the **vector space** of 2-forms

3 Elemenentary 2-forms in $\mathbb{R}^3$

- $dy \wedge dz$
- $dx \wedge dz$
- $dx \wedge dy$

All other 2-forms are linear combinations of these basis elements

$p \in M$



M: Line

$$\omega : p \mapsto \underbrace{(\omega_p : T_pM \to \mathbb{R})} \tag{11}$$

Provided a point (p), and a vector from the tangent space of the Manifold (M) at that point (T_pM) a scalar is returned.
"Provide a point, and a function is returned, that function given a tangent vector to the manifold returns a scalar"

$\int^{b} \omega$ $\mathbb{R}$

k-forms

For a given $\mathbb{R}^n$ space there are $\binom{n}{k}$ elementary k-forms.
Each k-form can be written as:

$$(k) \qquad \omega = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} f_{i_1 \ldots i_k}(x)\, \underbrace{dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}}_{\text{Elem k-forms}} \tag{12}$$

The exterior derivative

$$d : \text{k-forms} \rightarrow \text{(k+1)-forms} \tag{13}$$

Given $\underbrace{\omega = \sum_{\forall I} f_I dx_I}$ (a k-form)

wedge product

$$d\omega = \sum_{\forall I} df_I \wedge dx_I \tag{14}$$

functions

Assume $\mathbb{R}^3$
d(0-form) $\rightarrow$ 1-form
$\omega_0 = f(x_1, x_2, x_2)$

$e_1 = dx_1$ $e_2 = dx_2$ $e_3 = dx_3$

$$df = \frac{\delta f}{\delta dx_1} dx_1 + \frac{\delta f}{\delta dx_2} dx_2 + \frac{\delta f}{\delta dx_3} dx_3 \tag{15}$$

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right)$$

The exterior derivative of 1-forms

Assume $\mathbb{R}^3$

$d(\text{1-form}) \to \text{2-form}$

$\omega_1 = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$

$$
\begin{aligned}
d\omega &= df_1 \wedge dx_1 + df_2 \wedge dx_2 + df_3 \wedge dx_3 \\
&= \left(\frac{\delta f_1}{\delta x_1} dx_1 + \frac{\delta f_1}{\delta x_2} dx_2 + \frac{\delta f_1}{\delta x_3} dx_3\right) \wedge dx_1 + (..) \wedge dx_2 + (..) \wedge dx_3 \\
&= \left(\frac{\delta f_3}{\delta x_2} - \frac{\delta f_2}{\delta x_3}\right) dx_2 \wedge dx_3 + \left(\frac{\delta f_1}{\delta x_3} - \frac{\delta f_3}{\delta x_1}\right) dx_3 \wedge dx_1 + \left(\frac{\delta f_2}{\delta x_1} - \frac{\delta f_1}{\delta x_2}\right) dx_1 \wedge dx_2
\end{aligned}
$$

The exterior derivative of 1-forms

d (2-form) $\to$ div

Assume $\mathbb{R}^3$

d(1-form) $\to$ 2-form

$\omega_1 = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$

$\nabla \to$ 1D grad

$\nabla \times F \to$ 2D curl

$\nabla \cdot F \to$ 3D div

$f_3 : F_z$ $\quad x_1 : x$

$f_2 : F_y$ $\quad x_2 : y$

$f_1 : F_x$ $\quad x_3 : z$

$$
\begin{aligned}
d\omega &= df_1 \wedge dx_1 + df_2 \wedge dx_2 + df_3 \wedge dx_3 \\
&= (\frac{\delta f_1}{\delta x_1} dx_1 + \frac{\delta f_1}{\delta x_2} dx_2 + \frac{\delta f_1}{\delta x_3} dx_3) \wedge dx_1 + (..) \wedge dx_2 + (..) \wedge dx_3 \\
&= (\frac{\delta f_3}{\delta x_2} - \frac{\delta f_2}{\delta x_3}) dx_2 \wedge dx_3 + (\frac{\delta f_1}{\delta x_3} - \frac{\delta f_3}{\delta x_1}) dx_3 \wedge dx_1 + (\frac{\delta f_2}{\delta x_1} - \frac{\delta f_1}{\delta x_2}) dx_1 \wedge dx_2
\end{aligned}
$$

Which is precisely the curl:

$$
\nabla \times \vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)
$$