20250616 Cool Math Kids CMK Group

Jon Bosque

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Koopman Operator Theory

Rocq Theorem Prover

Koopman Operator Theory References

- Igor Mezic ME203
- Modern Koopman Theory for Dynamical Systems

Autonomous system

$$\frac{dx}{dt} = \dot{x} = F(x), \quad x \in \mathbb{R}^n \tag{1}$$

Trajectories

$$x(t) = S(x_0, t) \tag{2}$$

Flow

$$S^t(x_0) = S(x_0, t) \tag{3}$$

Observables map from the state x to a vector of observations.
This observations are transformations of the original state.

$$g(t, x_0) = g(S^t(x_0)) \tag{4}$$

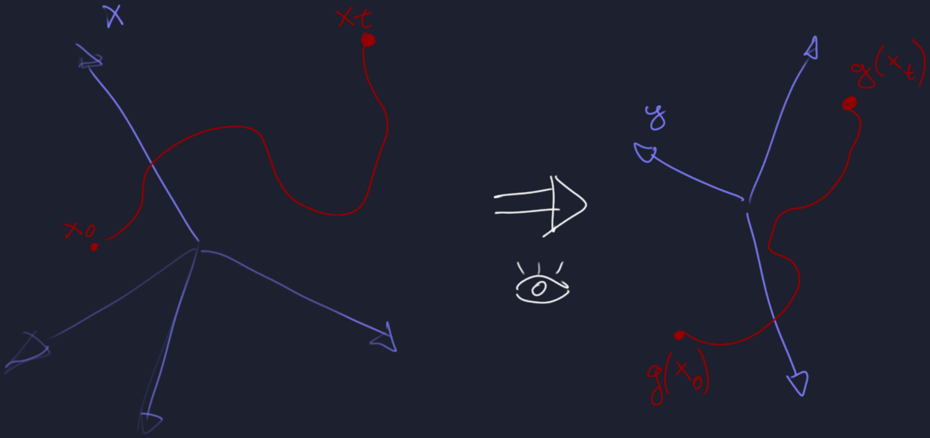


Figure: System Observation

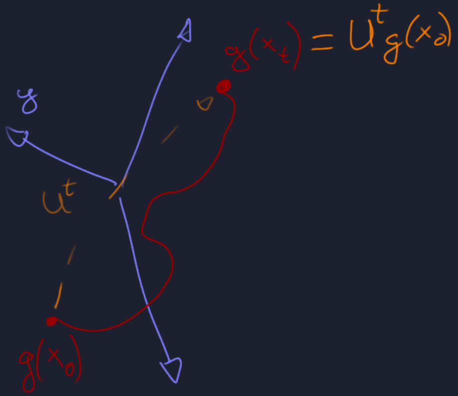


Figure: Application of the Koopman Operator

The Koopman Operator evolves observations of the system in time from a specific observation
"Dynamics on observation space"

$$U^t g(x_0) = g(S^t(x_0)) \tag{5}$$

The Koopman operator is a **linear** infinite dimensional operator, even for non-linear systems.

Spectral Analysis

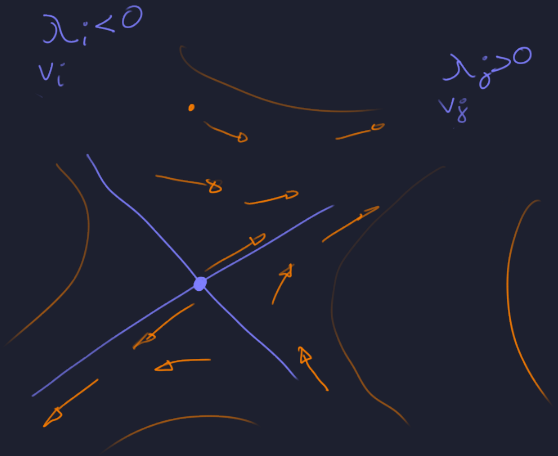
Given a Linear Dynamical system

$$\dot{x} = Ax \tag{6}$$

The eigen-decomposition of A determines the characteristic dynamics on each of the principal components of the system.

- $Re[\lambda_i] < 0$: Stable System on direction i
- $Re[\lambda_i] > 0$: Unstable System on direction i
- $Im[\lambda_i] \neq 0$: Oscillatory behavior on direction i

Phase Plane example



Koopman Operator Decomposition

In a Linear System, the eigenvectors of a system fullfil

$$Av = \lambda v \tag{7}$$

In the case of the Koopman operator, the eigenfunctions of a system, ϕ, fullfil

$$U\phi(x_k) = \lambda\phi(x_k) \tag{8}$$

Table: Conceptual mapping Linear → Non-linear

	Linear	**Non-linear**
Magnitude	λ	λ
Direction	$v \in R^n$	$\phi \in L^2(\Omega)$

1: The domain Ω in $L^2(\Omega)$ represents the set over which functions are defined. The functions must satisfy $\int_\Omega |f(x)|^2 dx < \infty$, f is square-integrable over the domain Ω.

Koopman Operator Realization

- Approximate finite dimensional representations of the Koopman Operator are realizable by the Dynamic Mode Decomposition algorithm
- Spectral analysis and other tools in linear system theory are applicable to this representations (eigenvalues+eigenfunctions)
- Dominant mode analysis
- Partition analysis

Rocq Theorem Prover References

- Software Foundations Vol1. Logical Foundations

Rocq Theorem Prover

An interactive theorem prover, and proof assistant.

Example